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ON THE DETECTION AND ESTIMATION PROBLEM FOR MULTIDIMENSIONAL GAUSSIAN RANDOM CHANNELS

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ABSTRACT

Two problems in statistical communication theory, the detection and estimation of intelligence-carrying signals are considered for the case of transmission through a multidimensional Gaussian random channel. Such a channel is characterized by the property that a deterministic input results in a set of received waveforms which are sample functions of Gaussian processes.

For the detection problem, a receiver is found which operates on the set of received waveforms and gives as outputs, voltages proportional to the logarithm of the likelihood functions of the possible transmitted signals. For the estimation problem, it is assumed that the intelligence-carrying signal is itself a sample function of a Gaussian process and a mathematical description is presented of a receiver which has as its output the maximum a posteriori estimate of this signal. Examples are presented in which optimum receivers are found for both the detection and estimation problem.

PUBLICATION REVIEW

This report has been reviewed and is approved.

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ON THE DETECTION AND ESTIMATION PROBLEM FOR MULTIDIMENSIONAL GAUSSIAN RANDOM CHANNELS*

INTRODUCTION

In communication systems where information is to be transmitted from a source to a receiver, the intelligence-carrying signal is always partly distorted and masked in transmission. A major problem in the field of statistical communication theory is to determine a method to process the received waveforms in order to recover from them as much of the intelligence as possible.

In certain communication systems the receiver must determine if a certain signal (or more generally, which of given set of signals) is present in a received waveform. An example of a system of this type is a digital transmission system where a predetermined set of signals is chosen as the signaling alphabet. In other communication systems the receiver is required to operate on the received waveforms in order to produce an estimate of the waveshape of the intelligence-carrying signal. An analogue voice communication circuit might serve as an example of such a system. The processing of the received waveforms in these two types of systems will be called, respectively, detection and estimation.

Much of the original work on the detection[1-4] and estimation[5-8] of signals has been concerned with the processing of a single received waveform which consists of the intelligence-carrying signal perturbed by additive Gaussian noise. A wide variety of practical problems, however, cannot be satisfactorily described by such a model. In the first place, for most practical situations, distortions other than additive Gaussian noise are present in the channel. Secondly, in many communication systems, the receiver is required to simultaneously process multiple received waveforms rather than the single received waveform assumed in the simplified model. In this report, an attempt will be made to increase the generality of the mathematical model so as to broaden the class of problems to which the theory applies. For other treatments of this problem, the reader is referred to the work of Turin,[9-10] Price,[11-12] Kailath[13] and others.[14-15] The portions of this study concerned with the detection problem will be found to overlap, to some degree, the work of Kailath.

FORMULATION

In this report, the problems of the detection and estimation of signals will be considered for the situation where the signals have been transmitted through a multidimensional Gaussian random channel. For the purposes of this report, a multidimensional Gaussian random channel will be defined as a transmission channel which, having been excited by any deterministic signal, results in a set of received waveforms which are sample functions of Gaussian random processes. Thus, the term multidimensional refers to the multiplicity of the received waveforms and the term Gaussian describes the joint statistics of these waveforms.

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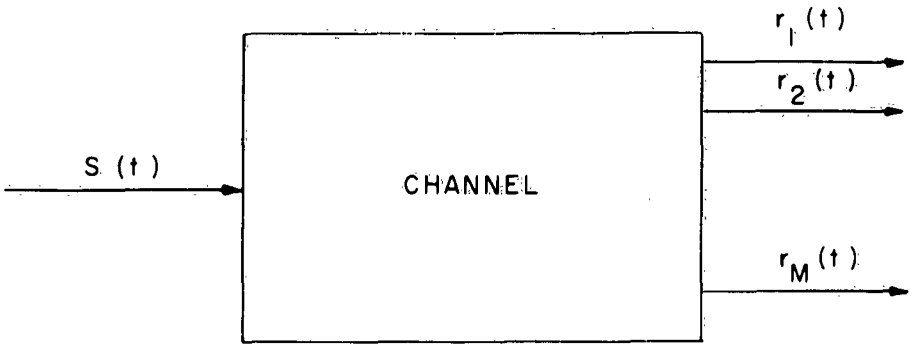

Figure 1. Multidimensional Transmission Channel

In general, these waveforms may be correlated with one another. It is to be noted that the channel is not required to be linear.

Such a channel can be interpreted in terms of the block diagram shown in Figure 1. Here, the intelligence-carrying signal, $S(t)$, enters the multidimensional transmission channel and results in the set of M received waveforms $r_1(t), r_2(t), \dots, r_M(t)$. The multidimensional channel will be said to be Gaussian if the conditional probability of this set of received waveforms, given a particular $S(t)$, is a jointly Gaussian probability density function.

It should be emphasized that the ensemble statistics and not the time statistics of the received waveforms are required to be Gaussian. For instance, in the case of a single received waveform composed of a deterministic signal in additive Gaussian noise, the ensemble statistics of the received waveform are Gaussian while in most cases the time statistics are not. Thus, the classical problem of the detection of a known signal in additive Gaussian noise is seen to be a special case of this model.

Since there are inherent differences in the formulation of the detection and estimation problems, it will be convenient to discuss each of these problems separately. For the detection problem, referring to Figure 2, it is assumed that a set of M received waveforms $r_1(t), r_2(t), \dots, r_M(t)$, are available at the receiver over an interval $(0, T)$. These wave forms are the result of one out of N signals $S^{(i)}(t)$, $i = 1, 2, \dots, N$, chosen by the

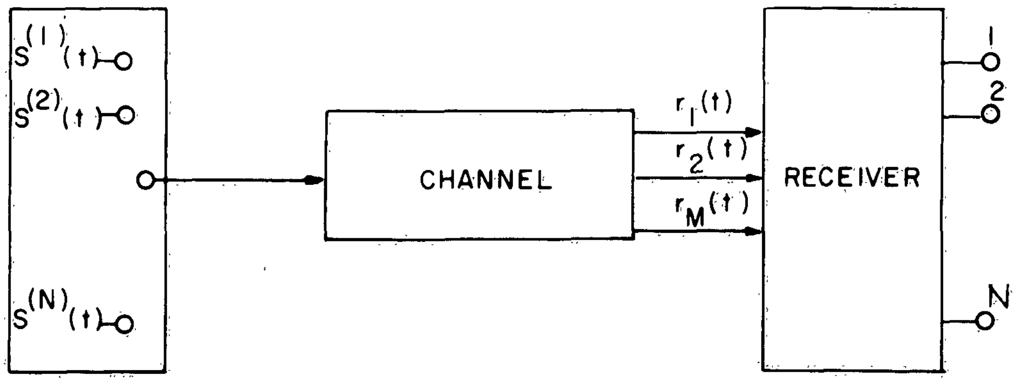

Figure 2. Detection Problem

transmitter with a priori probability $p(S^{(i)})$, entering the transmission channel. The M received waveforms are to be processed by the receiver so as to determine which of the N signals was actually chosen by the transmitter. A complete statistical description of the set of received waveforms is assumed available to the receiver as well as the a priori probabilities of the transmitted signals.

A slight variation of this detection problem which arises in radar applications is shown in Figure 3. Here the transmitter always transmits the same deterministic signal but the

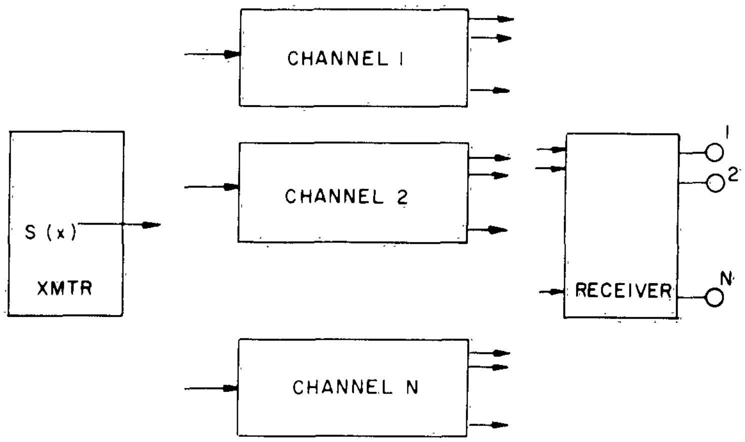

Figure 3. Detection Problem for Radar

signal travels through one of N transmission channels with some given a priori probability. These N channels have different transmission characteristics; for example, in the case of range radar, different delays would be assigned to the different channels. The receiver is to decide on the basis of a $(0, T)$ sample of the M received waveforms $r_1(t)$, $r_2(t), \ldots, r_M(t)$, through which of the N channels the signal passed. As in the previous case, it is assumed that the statistics of the received waveforms and the a priori probabilities of the channels are known to the receiver.*

For either configuration of the detection problem, the situation is one of multiple-alternative hypothesis testing. That is, given the set of received waveforms, the receiver is to choose among N mutually exclusive hypotheses. For the radar problem, the ith hypothesis, denoted H_i, corresponds to the situation where the signal was transmitted through the ith channel, while for the communications problem, H_i indicates that $S^{(i)}(t)$ was sent by the transmitter.

A short discussion of multiple-alternative hypothesis testing follows. The reader is referred to the references for amplification of these remarks. Woodward[1] has observed that a receiver which presents the a posteriori probabilities of the hypotheses H_i, $i = 1,2,\ldots, N$, tells as much as it is possible to know about these hypotheses from a knowledge of the received waveforms. A Bayes' decision rule,[16-17] which minimizes the average cost of

* Because of the repetitive nature of the radar transmitted pulse, the a priori probabilities for the hypotheses during the interval $(0, T)$ can sometimes be taken as the a posteriori probabilities for the hypotheses during the preceding interval.[1]

the decisions, has been shown to depend on the relative magnitudes of these a posteriori probabilities. Another decision rule, the Neyman-Pearson criterion,[18] is useful in testing between two hypotheses where it is not convenient to assign a priori probabilities to these hypotheses. A generalization of this criterion to multiple-alternative hypothesis testing has shown[14] that the decision can be based on the relative magnitudes of the likelihood functions* for the hypotheses. If the a priori probabilities of these hypotheses are known, the a posteriori probabilities can be obtained from the likelihood functions by Bayes' rule. Thus, all of the above criteria for multiple-alternative hypothesis testing depend on the relative magnitudes of the likelihood functions or the a posteriori probabilities (which can be obtained from the likelihood functions). Since the logarithm is a monotonic increasing function of its argument, decisions can be made using any of these criteria on the basis of quantities which are proportional to the logarithm of the N likelihood functions.

The optimum detector can then be divided into two parts. The first part operates on the received waveforms and results in outputs which are proportional to the logarithm of the likelihood functions. The second part weights these outputs in accordance with some criterion in order to make a decision between the hypotheses. In this report, systems will be considered only for instrumenting the first part of this detector.

The formulation of the estimation problem will next be considered. It is assumed that the intelligence-carrying signal to be estimated is itself a sample function of a Gaussian random process. This signal enters the transmission channel, resulting in the M received waveforms $r_1(t), r_2(t), \dots, r_M(t)$. This problem can be interpreted in terms of the block diagram shown in Figure 1, but now $S(t)$ is one out of an infinite number of signals, these signals being sample functions of a Gaussian process. It is desired to operate on the set of received waveforms, received during the interval $(0, T)$, to obtain an optimum estimate of the signal $S(t)$. The only restriction placed on the channel is that for any particular realization of the signal, the set of received waveforms are sample functions of Gaussian random processes; that is, we have a multidimensional Gaussian random channel. For simplicity it is assumed that the intelligence-carrying signal is statistically independent of the statistics of the channel.

It should be noted that many operations normally associated with the transmitter may be considered as part of this Gaussian random channel. For instance, if the intelligence-carrying signal is modulated before transmission, this modulating process (linear or nonlinear) can be included in the channel operation. Thus, the estimation of modulated signals corrupted by additive Gaussian noise, a problem previously considered by Youla,[6] would be a special case of this problem.

The optimization criterion used in this report is that of maximum a posteriori probability.[5] This method of estimation requires the receiver to choose which realization of the intelligence-carrying signal was most likely sent, given the set of received waveforms $r_1(t), r_2(t), \dots, r_M(t)$. More explicitly, the receiver is to choose the realization of the intelligence-carrying signal which maximizes the a posteriori probability. This optimization

* The term likelihood function is used to describe the conditional probability of the effect (the received waveforms) given the cause (the transmitted signal).[1]

criterion has also been referred to in the literature as maximum likelihood estimation[6] and a posteriori most probable estimation.[8]

It should be realized that the output of the receiver in the estimation problem is an analogue signal while the receiver for the detection problem has outputs which are voltages that represent the conditional probabilities for each of the hypotheses.

A MULTIDIMENSIONAL ORTHOGONAL EXPANSION

Since both the detection and estimation problems have been phrased in terms of multiple received waveforms, it is convenient to introduce vector and matrix notation. Consider the M waveforms, $n_1(t)$, $n_2(t)$, ... , $n_M(t)$, each of which is a sample function of a random process. In general these waveforms may be correlated with one another. These M waveforms can be denoted by the M dimensional vector $\overline{n}(t)$, the jth component of this vector, written $n_j(t)$, representing the jth waveform. Let the ensemble means of these waveforms be written as the vector $\overline{m}(t)$. The jth component of this vector mean is then given by the expression,

$$m_j(t) = E\{n_j(t)\}, \tag{1}$$

where $E\{\ \}$ denotes the statistical expectation of the term in the brackets.

The second order statistics of these M received waveforms can be described by a covariance function matrix, written $\underset{\sim}{R}(t,x)$. The element in the jth row and kth column of this matrix is defined by the expression

$$R_{jk}(t,x) = E\{[r_j(t) - m_j(t)]\ [r_k(x) - m_k(x)]\}. \tag{2}$$

Note that if the waveforms are sample functions of Gaussian random processes the mean vector $\overline{m}(t)$ and the covariance function matrix $\underset{\sim}{R}(t,x)$ give a complete statistical description of the waveforms.

In expressions involving vectors and matrices, the following notation will be used. The dot product of two M dimensional vectors $\overline{A}(t)$ and $\overline{B}(t)$, written $\overline{A}(t) \cdot \overline{B}(t)$, is given by the expression

$$\overline{A}(t) \cdot \overline{B}(t) \overset{\Delta}{=} \sum_{i=1}^{M} A_i(t)\, B_i(t). \tag{3}$$

The trace of the $M \times M$ matrix $\underset{\sim}{R}(t,x)$ will be written $Tr\,[\underset{\sim}{R}(t,x)]$ and is defined as

$$Tr\,[\underset{\sim}{R}(t,x)] \overset{\Delta}{=} \sum_{i=1}^{M} R_{ii}(t,x). \tag{4}$$

Using this notation, a summary will be presented of a multidimensional expansion that is useful in the solution of both the estimation and detection problems. This expansion is a generalization[19] of the Karhunen-Loève[20-21] expansion to vector stochastic processes. The proof of the following statements regarding this expansion can be found in the literature.[14-15]

Let the M-component vector stochastic process $\overline{n}(t)$ with mean vector $\overline{m}(t)$ and covariance function matrix $\underset{\sim}{R}(t,x)$ be represented in the interval $(0, T)$ by the expression

$$\bar{n}(t) = \bar{m}(t) + \sum_{j=1}^{\infty} N_j \, \bar{\theta}_j(t), \qquad 0 \leq t \leq T. \tag{5}$$

If the vectors $\bar{\theta}_j(t)$ are chosen as the eigenvectors of the matrix integral equation

$$\bar{\theta}_j(t) = \lambda_j \int_0^T \underset{\sim}{R}(t,x) \, \bar{\theta}_j(x) \, dx, \qquad 0 \leq t \leq T, \tag{6}$$

and if these vectors are orthogonal and normalized such that

$$\int_0^T \bar{\theta}_j(t) \cdot \bar{\theta}_k(t) \, dt = \delta_{jk}, \tag{7}$$

then the coefficients N_j satisfy the following equations:

$$E\{N_j\} = 0, \tag{8}$$

$$E\{N_j N_k\} = \frac{\delta_{jk}}{\lambda_j} \tag{9}$$

Here δ_{jk} is the familiar Kronecker delta defined as

$$\delta_{jk} \stackrel{\Delta}{=} \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} . \tag{10}$$

From equations (5) and (7), the coefficients N_j are given by the expression

$$N_j = \int_0^T \left[\bar{n}(t) - \bar{m}(t)\right] \cdot \bar{\theta}_j(t) \, dt. \tag{11}$$

Note that subscripts have been used both for indexing eigenvectors of integral equations and also for denoting components of vectors. Then, the kth component of the vector $\bar{\theta}_j(t)$ will be written $\theta_{j,k}(t)$, the comma between the subscripts differentiating this notation from the double subscript notation used for elements of matrices.

From a generalization of Mercer's theorem, the i-jth element of the covariance function matrix can be written

$$R_{ij}(t,x) = \sum_{k=1}^{\infty} \frac{\theta_{k,i}(t) \, \theta_{k,j}(x)}{\lambda_k} . \tag{12}$$

If an inverse covariance function matrix, $\underset{\sim}{Q}(x,z)$, is defined as

$$\int_0^T \underset{\sim}{R}(t,x) \, \underset{\sim}{Q}(x,z) \, dx = \delta(t-z) \, \underset{\sim}{1}, \tag{13}$$

where $\underset{\sim}{1}$ is the unit matrix and $\delta(t-z)$ is the Dirac delta-function, the $i-j$th element of this matrix can be expanded in the form

$$Q_{ij}(x,z) = \sum_{k=1}^{\infty} \lambda_k \, \theta_{k,i}(x) \, \theta_{k,j}(z). \tag{14}$$

Let us now assume that the components of $\overline{n}(t)$ are sample functions of Gaussian random processes. Then, the coefficients N_j are Gaussian random variables and from equation (9) are uncorrelated and thus statistically independent. The probability density for the noise vector $\overline{n}(t)$, which is the joint probability of the set of coefficients N_j, can then be written*

$$p(\overline{n}) = \prod_{k=1}^{\infty} \left[\sqrt{\frac{\lambda_k}{2\pi}} \exp\left(-\frac{1}{2} \lambda_k N_k^2 \right) \right] \tag{15}$$

or

$$p(\overline{n}) = \prod_{k=1}^{\infty} \left[\sqrt{\frac{\lambda_k}{2\pi}} \right] \exp\left(-\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k N_k^2 \right) . \tag{16}$$

From equation (11), the series $\sum_{k=1}^{\infty} \lambda_k N_k^2$ can be written

$$\sum_{k=1}^{\infty} \lambda_k N_k^2 = \int_0^T \int_0^T \sum_{i=1}^{M} \sum_{j=1}^{M} [n_i(t) - m_i(t)] \, [n_j(x) - m_j(x)] \sum_{k=1}^{\infty} \lambda_k \theta_{k,i}(t) \theta_{k,j}(x) \, dt \, dx. \tag{17}$$

Referring to equation (14), and using vector and matrix notation, this series becomes

$$\sum_{k=1}^{\infty} \lambda_k N_k^2 = \int_0^T \int_0^T [\overline{n}(t) - \overline{m}(t)] \cdot \underset{\sim}{Q}(t,x) \; [\overline{n}(x) - \overline{m}(x)] \, dt \, dx. \tag{18}$$

Since $Q_{ij}(t,x) = Q_{ji}(x,t)$ it can be shown that equation (18) can be rewritten as

$$\sum_{k=1}^{\infty} \lambda_k N_k^2 = 2 \int_0^T [\overline{n}(t) - \overline{m}(t)] \cdot \left[\int_0^t \underset{\sim}{Q}(t,x) \; [\overline{n}(x) - \overline{m}(x)] \, dx \right] dt. \tag{19}$$

From equation (16), the natural logarithm of the probability density for the noise vector $\overline{n}(t)$ can be expressed as

$$\log p(\overline{n}) = K + \frac{1}{2} \sum_{k=1}^{\infty} \log \lambda_k - \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k N_k^2, \tag{20}$$

where K is a constant. Combining equations (19) and (20) we can write

$$\log p(\overline{n}) = K + \frac{1}{2} \sum_{k=1}^{\infty} \log \lambda_k - \int_0^T [\overline{n}(t) - \overline{m}(t)] \cdot \left[\int_0^t \underset{\sim}{Q}(t,x) \; [\overline{n}(x) - \overline{m}(x)] \, dx \right] dt. \tag{21}$$

---

* It is realized that some difficulties are encountered in the convergence of equations (15) and (16). Such difficulties can be avoided by using a finite number of terms in the series or by considering the ratio $\frac{p(\overline{n})}{p(\overline{n}_0)}$ where $\overline{n}_0(t)$ is a reference noise process. These difficulties will be ignored in this report but the reader should understand that one of these two procedures can always be used to rigorize the work to follow.

In the work to follow, the set of received waveforms $\bar{r}(t)$ will be represented in the interval $(0,T)$ by a multidimensional orthogonal expansion with uncorrelated coefficients similar to that given for $\bar{n}(t)$ and described in equations (5) through (11). In the most general case, the mean vector and the covariance function matrix may depend upon which particular realization of the intelligence-carrying signal was transmitted (for the estimation problem) or which hypothesis was true (for the detection problem). Note that if the covariance function matrix depends upon these conditions so will the eigenvectors and eigenvalues of the matrix integral equation having the covariance function matrix as its kernel. The superscript i will be used with functions in the detection problem to indicate that hypothesis H_i was assumed true in writing those functions. For the estimation problem, the superscript s on the functions will indicate that a particular realization of the intelligence-carrying signal was assumed.

CALCULATION OF THE LIKELIHOOD FUNCTIONS (DETECTION PROBLEM)

In this section systems will be derived which operate on the set of received waveforms and result in outputs which are proportional to the logarithm of the N likelihood functions corresponding to the N possible hypotheses. The mathematics needed to derive expressions describing these systems have been discussed in the previous section. By analogy with this work, let us consider that the set of received waveforms, $\bar{r}(t)$, are expanded in a multidimensional orthogonal expansion, under the assumption that hypotheses H_i is correct. Then, following the development given in the last section, an equation is obtained for the logarithm of the conditional probability density of $\bar{r}(t)$, assuming the truth of hypothesis H_i. This conditional probability density is just the likelihood function for hypothesis H_i and, referring to equation (21), is given by the expression

$$\log p(\bar{r}|H_i) = K + \frac{1}{2}\sum_{k=1}^{\infty} \log \lambda_k^{(i)} - \int_0^T [\bar{r}(t) - \bar{m}^{(i)}(t)] \cdot [\int_0^t Q^{(i)}(t,x)\,[\bar{r}(x) - \bar{m}^{(i)}(x)]\,dx]\,dt. \tag{22}$$

Here, $\bar{m}^{(i)}(t)$ is the vector mean of $r(t)$, assuming hypotheses H_i; $\underset{\sim}{Q}^{(i)}(t,x)$ is the inverse covariance function matrix of $\bar{r}(t)$, assuming hypothesis H_i; $\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)} \ldots$ are the eigenvalues of a matrix integral equation having the covariance function matrix $\underset{\sim}{R}^{(i)}(t,x)$ as its kernel; and K is a constant.

To emphasize the meaning of the notation, the jth component of the vector mean $\bar{m}^{(i)}(t)$ is given by the expression

$$m_j^{(i)}(t) = E\{r_j(t) \mid H_i\}, \tag{23}$$

where $E\{r_j(t)\mid H_i\}$ is the conditional expectation of $r_j(t)$, assuming hypothesis H_i. Also the j-kth element of the covariance function matrix $\underset{\sim}{R}^{(i)}(t,x)$ is given as

$$R_{jk}^{(i)}(t,x) = E\{[r_j(t) - m_j^{(i)}(t)]\,[r_k(x) - m_k^{(i)}(x)] \mid H_i\}. \tag{24}$$

The receiver is then to calculate the functions log $p(\overline{r}\,|\,H_i)$ for $i = 1, 2, \dots, N$. The assumption has been made that the receiver has a complete statistical description of the set of received waveforms. Since the received waveforms are Gaussian random processes, the receiver must have available the components of the vector mean $\overline{m}^{(i)}(t)$ and the elements of covariance function matrix $\underset{\sim}{R}^{(i)}(t,x)$ for all values of i.

As an illustrative example, consider the transmission channel shown in Figure 4 where the additive noise $n_1(t)$ and multiplicative noise $n_2(t)$ are Gaussian, have zero mean and are uncorrelated with one another. Then the received waveforms are Gaussian processes

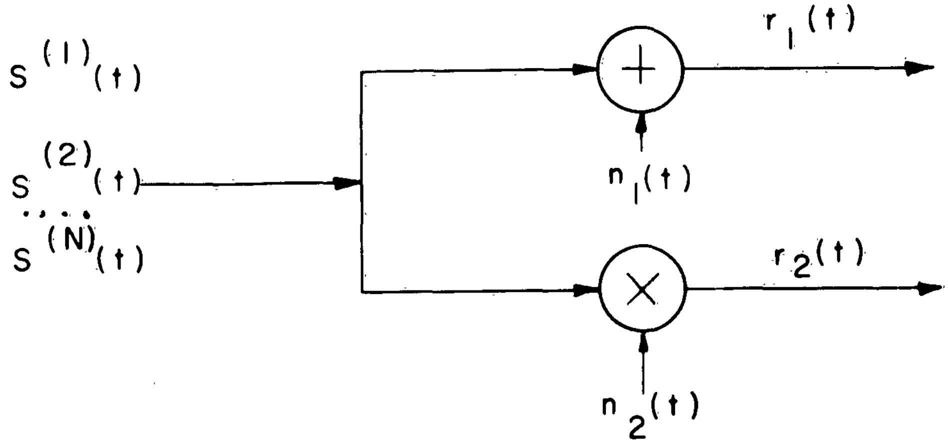

Figure 4. Example

with means and variances given by the equations

$$m_1^{(i)}(t) = S^{(i)}(t) \tag{25}$$

$$m_2(t) = 0 \tag{26}$$

$$R_{11}(t,x) = E\,\{n_1(t)\; n_1(x)\} \tag{27}$$

$$R_{22}^{(i)}(t,x) = E\,\{n_2(t)\; n_2(x)\}\; S^{(i)}(t)\; S^{(i)}(x) \tag{28}$$

$$R_{12}^{(i)}(t,x) = R_{21}^{(i)}(x,t) = E\,\{n_1(t)\,n_2(x)\}\; S^{(i)}(x). \tag{29}$$

The receiver must then have knowledge of the N signals $S^{(1)}(t), S^{(2)}(t), \dots, S^{(N)}(t)$, and the covariances $E\,\{n_1(t)\; n_1(x)\}$, $E\,\{n_2(t)\; n_2(x)\}$, and $E\,\{n_1(t)\; n_2(x)\}$.

Let us now return to equation (22) and investigate systems which will calculate these functions. Defining the quantities $B^{(i)}$ and $C^{(i)}(\overline{r})$ as

$$B^{(i)} \triangleq \frac{1}{2} \sum_{k=1}^{\infty} \log \lambda_k^{(i)}, \tag{30}$$

$$C^{(i)}(\bar{r}) \triangleq -\int_0^T [\bar{r}(t) - \bar{m}^{(i)}(t)] \cdot [\int_0^t \underset{\sim}{Q}^{(i)}(t,x) \quad [\bar{r}(x) - \bar{m}^{(i)}(x)]\, dx\,]\, dt, \tag{31}$$

we can write

$$\log p(\bar{r}|H_i) = K + B^{(i)} + C^{(i)}(\bar{r}). \tag{32}$$

The first term, the constant K, is independent of the index i and thus can be ignored. The second term, $B^{(i)}$, will in general be different for different values of the index i. However, it does not represent an operation on the received waveforms and can be calculated prior to communicating. It should be noted that if the covariance function matrix $\underset{\sim}{R}^{(i)}(t,x)$ is independent of the index i, then $B^{(i)}$ would also be independent of the index i and could also be ignored. The $C^{(i)}(\bar{r})$ is a function that must be calculated from the M received waveforms. Rewriting equation (31) in terms of the components and elements of the vectors and matrices we have

$$C^{(i)}(\bar{r}) = \int_0^T \sum_{j=1}^{M} \sum_{k=1}^{M} [r_j(t) - m_j^{(i)}(t)] \int_0^t Q_{jk}^{(i)}(t,x) \; [r_k(x) - m_k^{(i)}(x)] \; dx\, dt, \tag{33}$$

which explicitly displays the operations necessary to calculate $C^{(i)}(\bar{r})$.

A system for calculating the function $\log p(\bar{r}|H_i)$ for the case of two received waveforms is shown in Figure 5. In this diagram, the blocks labeled $Q_{ij}(t,x)$ are linear filters, $Q_{ij}(t,x)$ being the response at time t of the filter to a unit impulse applied at time x. The complete receiver, of course, must contain M of these systems, one for each possible hypothesis. A complete receiver is shown in Figure 6 for the single input case.

A receiver which has outputs which are proportional to the a posteriori probabilities of the hypotheses instead of the likelihood functions can be obtained by a slight modification of the previous receiver. From Bayes' rule, the a posteriori probability of H_i, written $p(H_i|\bar{r})$, is given in terms of the likelihood function $p(\bar{r}|H_i)$ and the a priori probability $p(H_i)$ as

$$p(H_i|\bar{r}) = k \; p(\bar{r}|H_i)\; p(H_i), \tag{34}$$

where k is a constant. Taking logarithms and referring to equation (32), it is seen that the logarithm of the a posteriori probability for H_i can be written

$$\log p(H_i|\bar{r}) = (K + k) + (B^{(i)} + \log p(H_i)) + C^{(i)}(\bar{r}). \tag{35}$$

Thus, the same basic receiver can be used to obtain the functions $\log p(H_i|\bar{r})$ except for the bias term which is changed by the addition of the term $\log p(H_i)$. If all hypotheses have equal a priori probabilities, no modification of the basic receiver is required.

Configurations for obtaining the likelihood functions (and a posteriori probabilities) can be devised other than the one described in equations (30) through (33) and shown in

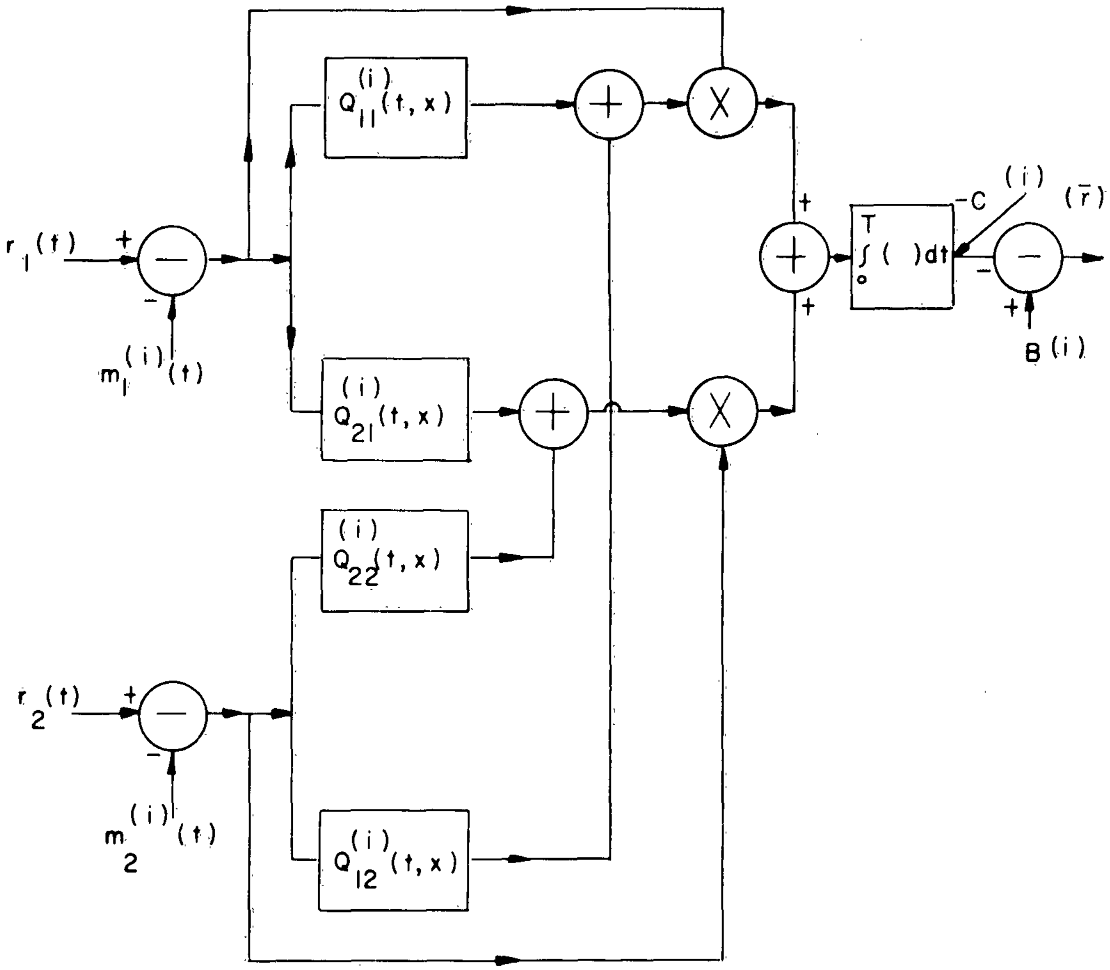

Figure 5. Calculation of $\log p(\bar{r}|H_i)$ for Two Received Waveforms

Figures (5) and (6). For instance, $\log p(\bar{r}|H_i)$ can be written,

$$\log p(\bar{r}|H_i) = K + D^{(i)} + E^{(i)}(\bar{r}), \tag{36}$$

where $L^{(i)}$ a bias term which can be calculated before transmission, is defined as

$$D^{(i)} \triangleq \frac{1}{2}\sum_{k=1}^{\infty} \log \lambda_k^{(i)} - \frac{1}{2}\int_0^T\int_0^T \bar{m}^{(i)}(t) \cdot \underset{\sim}{Q}^{(i)}(t,x)\, \bar{m}^{(i)}(x)\; dx\; dt \tag{37}$$

and $E^{(i)}(\bar{r})$, a term which is calculated from the received waveforms, is given as

$$E^{(i)}(\bar{r}) \triangleq \int_0^T\int_0^T [\bar{m}^{(i)}(t) - \frac{1}{2}\bar{r}(t)] \cdot \underset{\sim}{Q}^{(i)}(t,x)\, \bar{r}(x)\; dx\; dt. \tag{38}$$

Note that if the inverse covariance function matrix is independent of the index i,

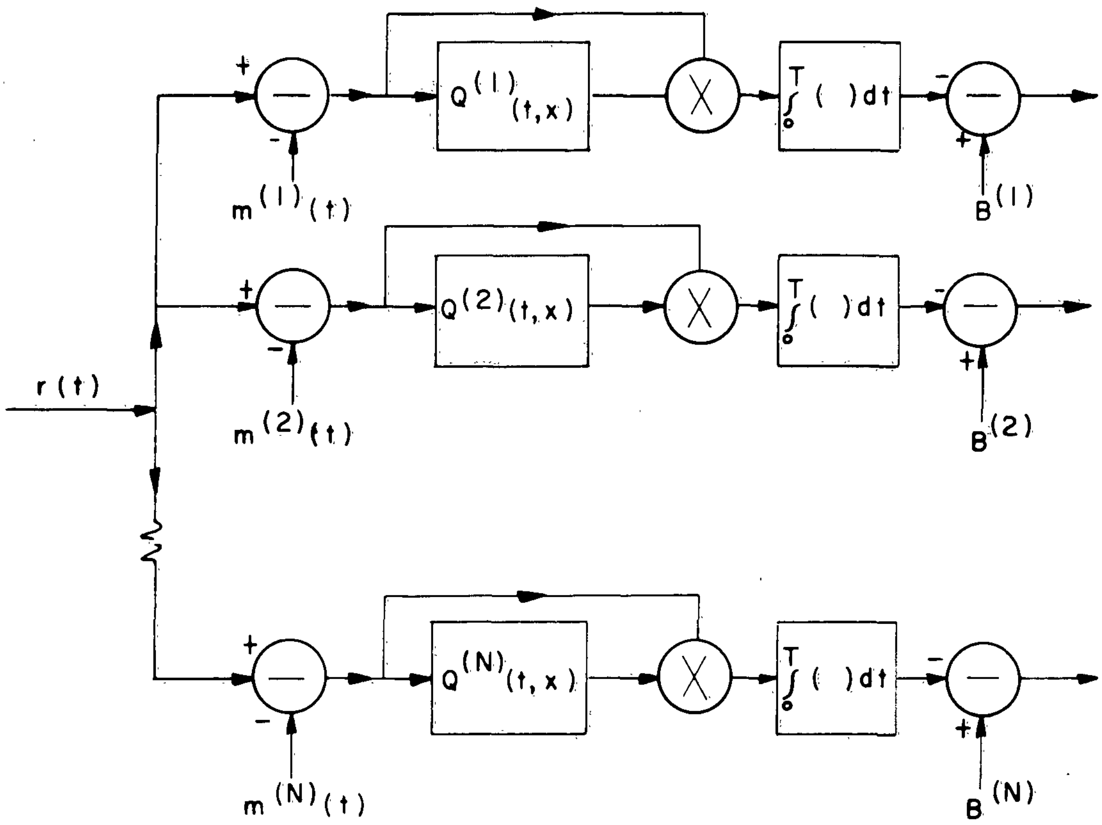

Figure 6. Complete Receiver - Nondiversity Case

the term $-1/2 \int_0^T \int_0^T \overline{r}(t) \cdot \underset{\sim}{Q}(t,x)\, \overline{r}(x)\, dx\, dt$ will be the same for all hypotheses and thus can be neglected.

An example illustrating some of these ideas for the case of a single received waveform is shown in Figure 7 Here, the transmitter chooses one of the signals $\sin \omega_i t$, $i = 1, 2, \ldots N$, and transmits it during the interval $(0, T)$. The multiplicative noises $m_1(t)$ and $m_2(t)$ are assumed to be uncorrelated, stationary, Gaussian processes with zero means and identical covariances $R_m(t-x)$,

$$R_m(t-x) = E\{m_1(t)\ m_1(x)\} = E\{m_2(t)\ m_2(x)\}. \tag{39}$$

The additive noise $n(t)$, which is uncorrelated with the multiplicative noises, is assumed to be a stationary Gaussian process with zero mean and covariance $R_n(t-x)$. The block denoted 90 degrees is assumed to be a 90 degree phase shifter; that is, a nonphysically realizable linear filter whose output is the Hilbert transform of its input.

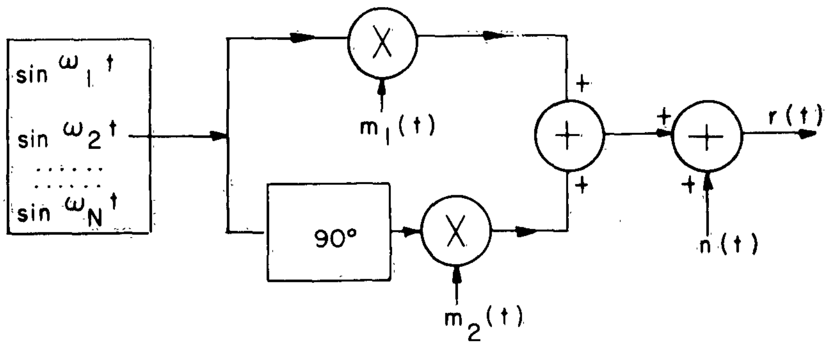

Figure 7. FSK Transmission through a Fading Channel

If the keying interval T is large enough $(T >> \frac{1}{\omega_i}$ for all i) the output of the 90 degree phase shifter will be cos $\omega_i t$ and the received waveform can be written

$$r(t) = n(t) + m_1(t) \sin \omega_i t + m_2(t) \cos \omega_i t, \tag{40}$$

or

$$r(t) = n(t) + R(t) \sin [\omega_i t + \phi(t)], \tag{41}$$

where $R(t)$ is Rayleigh and $\phi(t)$ is uniformly distributed $(-\pi, \pi)$. This model, which has been previously considered by Price,[11] then represents a narrow-band Rayleigh fading signal with additive Gaussian noise.

The mean of $r(t)$ is identically equal to zero for all H_i and the covariance $R^{(i)}(t,x)$ can be written as the difference of the two variables t and x and is given by the expression

$$R^{(i)}(t-x) = R_n(t-x) + R_m(t-x) \cos \omega_i(t-x). \tag{42}$$

Referring to equations (30) through (32) it can be shown that the bias term $B^{(i)}$ is the same for all hypotheses. The optimum receiver is then given by the block diagram shown in Figure 8, where the impulse responses $h^{(i)}(x,z)$ of the time varying linear filters are the solution to the integral equation,

$$\int_0^T [R_n(t-x) + R_m(t-x) \cos \omega_i(t-x)] \, h^{(i)}(x,z) \, dx = \delta(t-z), \tag{43}$$

$$0 \le t, z \le T.$$

Ignoring the physical realizability of these filters and assuming that T is large enough so that Fourier transform techniques can be used, the transfer function of the ith filter is given approximately by the expression

$$H^{(i)}(\omega) = \frac{1}{\phi_n(\omega) + 1/2\,[\phi_m(\omega+\omega_i) + \phi_m(\omega-\omega_i)]} \,. \tag{44}$$

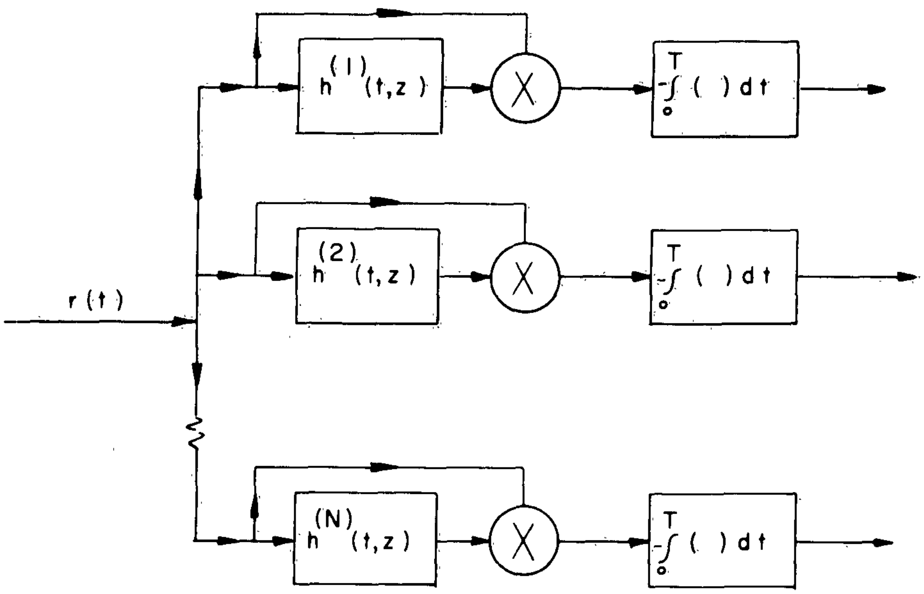

Figure 8. Optimum Receiver for Example Shown in Figure 7.

Here $\phi_n(\omega)$ and $\phi_m(\omega)$ are respectively the power spectra of the additive and multiplicative noises. Thus the filters are essentially band reject filters with the stop band centered at ω_i.

The outputs of the receiver shown in Figure 8 are, except for the omission of additive constants, the logarithm of the likelihood functions (or the logarithm of the a posteriori probabilities since the a priori probabilities were assumed equal). These functions are then used in making the decision.

As a second example consider the transmission system shown in Figure 9, where the intelligence-carrying signal $S(t)$ is one of the N signals $S^{(i)}(t)$, $i = 1, 2, \dots, N$. The two

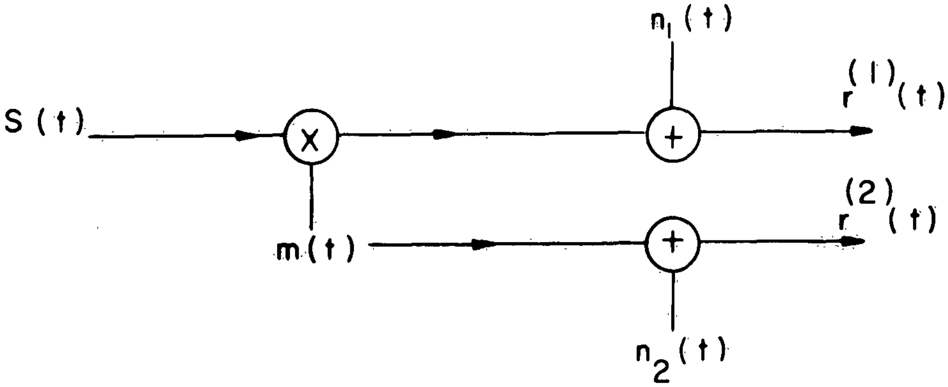

Figure 9. Transmitted Reference Communications System

received waveforms are given as

$$r^{(1)}(t) = S^{(i)}(t)\mathrm{m}(t) + n_1(t) \qquad 0 \leq t \leq T$$

$$r^{(2)}(t) = \mathrm{m}(t) + n_2(t) \,. \tag{45}$$

It is assumed that the noises $n_1(t)$, $n_2(t)$ and $\mathrm{m}(t)$ all are sample functions of independent Gaussian processes with zero means and covariance functions

$$E\,\{n_1(t)n_1(\tau)\} = \frac{N_{01}}{2}\,\delta(t-\tau), \tag{46}$$

$$E\,\{n_2(t)n_2(\tau)\} = \frac{N_{02}}{2}\,\delta(t-\tau), \tag{47}$$

$$E\,\{\mathrm{m}(t)\mathrm{m}(\tau)\} = \frac{\mathrm{M}_0}{2}\,\delta(t-\tau). \tag{48}$$

Such a situation would occur when a wide-band Gaussian noise $\mathrm{m}(t)$ is used as the carrier for transmitting digital signals and this noisy carrier is also transmitted over another channel for use by the receiver in the detection process.

The mean vector of $\overline{r}(t)$ is identically equal to zero and the covariance matrix $\underset{\sim}{R}^{(i)}(t,x)$ is given as

$$\underset{\sim}{R}^{(i)}(t,x) = \frac{\delta(t-x)}{2}\begin{bmatrix} \mathrm{M}_0\, S^{(i)}(t)S^{(i)}(x) + N_{01} & \mathrm{M}_0 S^{(i)}(t) \\ \mathrm{M}_0\, S^{(i)}(x) & \mathrm{M}_0 + N_{02} \end{bmatrix} \tag{49}$$

The inverse covariance function matrix $\underset{\sim}{Q}^{(i)}(t,x)$ is then

$$\underset{\sim}{Q}^{(i)}(t,x) = \frac{2\,\delta(t-x)}{N_{01}N_{02} + \mathrm{M}_0(N_{02}S^{(i)}(t)S^{(i)}(x) + N_{01})} \times$$

$$\begin{bmatrix} \mathrm{M}_0 + N_{02} & -\mathrm{M}_0 S^{(i)}(x) \\ -\mathrm{M}_0 S^{(i)}(t) & \mathrm{M}_0 S^{(i)}(t)S^{(i)}(x) + N_{01} \end{bmatrix} . \tag{50}$$

To further restrict the problem it is assumed that the transmitter is sending equiprobable binary digits, the two possible signals being a signal $S(t)$ and its negative, that is,

$$S^{(i)}(t) = \begin{cases} S(t) & i = 1 \\ -S(t) & i = 2. \end{cases} \tag{51}$$

In this case it can be shown that the bias term $B^{(i)}$ is the same for both hypotheses. The logarithm of the likelihood function is then given as

$$\log p(\bar{r} \mid H_i) = \begin{cases} K_1 + 4M_0 \int_0^T \dfrac{S(t) r^{(1)}(t)\, r^{(2)}(t)\, dt}{N_{01} N_{02} + M_0 (N_{02} S^2(t) + N_{01})} & i = 1 \\ \\ K_1 - 4M_0 \int_0^T \dfrac{S(t) r^{(1)}(t)\, r^{(2)}(t)\, dt}{N_{01} N_{02} + M_0 (N_{02} S^2(t) + N_{01})} & i = 2 \end{cases}, \tag{52}$$

where K_1 is a constant. Thus the optimum receiver calculates the quantity

$$\int_0^T \frac{S(t) r^{(1)}(t)\, r^{(2)}(t)\, dt}{N_{01} N_{02} + M_0 (N_{02} S^2(t) + N_{01})} .$$

Note that if $S(t)$ is a constant the optimum receiver just computes the short-time cross-correlation of the two received waveforms.

MAXIMUM A POSTERIORI PROBABILITY RECEIVER (ESTIMATION PROBLEM)

Let us now assume that the intelligence-carrying signal $S(t)$ is a sample function of a Gaussian random process with zero mean and covariance $R_s(t,x)$. The signal is transmitted through a multidimensional Gaussian random channel and the statistics of the signal and the channel are independent. It is desired to operate on the set of received waveforms, $\bar{r}(t)$, received during the interval $(0, T)$, in order to obtain the maximum posterior estimate of the signal $S(t)$. That is, the output of the receiver is to be that particular realization of $S(t)$ that maximizes the a posteriori probability $p(S \mid \bar{r})$. Since the logarithm is a monotonic increasing function of its argument, the realization of $S(t)$ which maximizes the a posteriori probability will also maximize the logarithm of the a posteriori probability.

By analogy with the work on detection, let us assume a particular realization for $S(t)$ and expand the set of received waveforms in the orthogonal expansion

$$\bar{r}(t) = \bar{m}^{(s)}(t) + \sum_{j=1}^{\infty} a_j^{(s)}\, \bar{\theta}_j^{(s)}(t), \qquad 0 \leq t \leq T, \tag{53}$$

where the $\bar{\theta}_j^{(s)}(t)$ are eigenvectors of the matrix integral equation

$$\lambda_j^{(s)} \int_0^T \underset{\sim}{R}^{(s)}(t,x)\, \overline{\theta}^{(s)}(x)\, dx = \overline{\theta}_j^{(s)}(t), \qquad 0 \le t \le T, \tag{54}$$

and are orthonormal; that is,

$$\int_0^T \overline{\theta}_j^{(s)}(t) \cdot \overline{\theta}_k^{(s)}(t)\, dt = \delta_{jk}. \tag{55}$$

In these equations $\overline{m}^{(s)}(t)$ is the vector mean and $\underset{\sim}{R}^{(s)}(t,x)$ is the covariance function matrix of the received waveforms, both assuming a particular realization of $S(t)$. Then, the logarithm of the conditional probability of $\overline{r}(t)$ given a particular $S(t)$ can be written

$$\log p(\overline{r}\,|S) = K_3 + \frac{1}{2}\sum_{j=1}^{\infty} \log \lambda_j^{(s)} - \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j^{(s)}\, (\alpha_j^{(s)})^2 . \tag{56}$$

Let us now expand the signal $S(t)$ in an ordinary Karhunen-Loève expansion. That is, let

$$S(t) = \sum_{j=1}^{\infty} \beta_j\, \psi_j(t), \tag{57}$$

where the $\psi_j(t)$ are the eigenfunctions of the integral equation

$$\gamma_j \int_0^T R_s(t,x)\, \psi_j(x)\, dx = \psi_j(t), \qquad 0 \le t \le T, \tag{58}$$

and are orthonormal,

$$\int_0^T \psi_j(t)\, \psi_k(t)\, dt = \delta_{jk}. \tag{59}$$

Then the logarithm of the a priori probability of $S(t)$ can be written as

$$\log p(S) = K_4 + \frac{1}{2}\sum_{j=1}^{\infty} \log \gamma_j - \frac{1}{2}\sum_{j=1}^{\infty} \gamma_j\, \beta_j^2 . \tag{60}$$

Using Bayes' rule and referring to equations (57) and (60), the logarithm of the a posteriori probability for $S(t)$ can be written

$$\log p(S\,|\overline{r}\,) = K_5 + \frac{1}{2}\sum_{j=1}^{\infty} \log \lambda_j^{(s)} + \frac{1}{2}\sum_{j=1}^{\infty} \log \gamma_j - \frac{1}{2}\sum_{j=1}^{\infty} \lambda_j^{(s)} (\alpha_j^{(s)})^2 - \frac{1}{2}\sum_{j=1}^{\infty} \gamma_j \beta_j^{\,2}. \tag{61}$$

The signal $S(t)$ which maximizes the logarithm of the a posteriori probability will be found by solving for the set of coefficients β_k that makes equation (61) a maximum. Thus, taking the partial derivative of the terms in equation (61) with respect to β_k and setting the result equal to zero results in the equation

$$\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\lambda_j^{(s)}}\frac{\partial\lambda_j^{(s)}}{\partial\beta_k}-\frac{1}{2}\sum_{j=1}^{\infty}\frac{\partial}{\partial\beta_k}\left[\lambda_j^{(s)}(\alpha_j^{(s)})^2\right]=\gamma_k\beta_k. \tag{62}$$

Multiplying equation (64) by the expression $\frac{\psi_k(t)}{\gamma_k}$ and summing over the index k results in the following expression for the estimate of $S(t)$,

$$\overset{*}{S}(t)=\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\overset{*}{\lambda}_j}\frac{\partial\overset{*}{\lambda}_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k}-\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\partial}{\partial\beta_k}\left[\overset{*}{\lambda}_j(\overset{*}{\alpha}_j)^2\right]\frac{\psi_k(t)}{\gamma_k}. \tag{63}$$

The notation $\overset{*}{S}(t)$ is used to indicate an estimate of $S(t)$. The functions with the superscript* also depend upon this estimate.

It is desired to find more convenient forms for the terms on the right hand side of equation (63). It is shown in the Appendix that the first term can be written as

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\overset{*}{\lambda}_j}\frac{\partial\overset{*}{\lambda}_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k}=\frac{1}{2}\int_0^T\int_0^T\sum_{k=1}^{\infty}Tr\left[\underset{\sim}{R}^*(x,y)\frac{\partial\underset{\sim}{Q}^*(y,x)}{\partial\beta_k}\right]\frac{\psi_k(t)}{\gamma_k}\,dx\,dy \tag{64}$$

and the second term becomes

$$-\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\partial}{\partial\beta_k}\left[\overset{*}{\lambda}_j(\overset{*}{\alpha}_j)^2\right]\frac{\psi_k(t)}{\gamma_k}=-\frac{1}{2}\sum_{k=1}^{\infty}\int_0^T\int_0^T\frac{\partial}{\partial\beta_k}\Big[[\bar{r}(x)-\bar{m}^*(x)].$$

$$\underset{\sim}{Q}^*(x,y)\,[\bar{r}(y)-\bar{m}^*(y)]\Big]\frac{\psi_k(t)}{\gamma_k}\,dx\,dy. \tag{65}$$

Then the optimum estimate $\overset{*}{S}(t)$ is given by the sum of these terms.

The inverse covariance function matrix $\underset{\sim}{Q}(y,x)$ will, in general, be a function of the intelligence-carrying signal. This functional dependence may occur through the appearance of terms which are explicit functions of $S(x)$ and $S(y)$. However, the matrix $\underset{\sim}{Q}(y,x)$ can depend implicitly on the intelligence-carrying signal; for example, $\underset{\sim}{Q}(y,x)$ could be a function of $\int_0^T S^2(z)\,dz$. A similar situation may occur for the vector mean $\bar{m}(x)$. If the intelligence-carrying signal appears in $\underset{\sim}{Q}(y,x)$ and $\bar{m}(x)$ only in terms which are explicit functions of $S(x)$ and $S(y)$, then it is shown in the Appendix that equation (64) becomes

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\overset{*}{\lambda}_j}\frac{\partial\overset{*}{\lambda}_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k}=\int_0^T\int_0^T R_s(t,x)\,Tr\left[\underset{\sim}{R}^*(x,y)\frac{\partial\underset{\sim}{Q}^*(y,x)}{\partial S(x)}\right]dx\,dy, \tag{66}$$

and equation (65) can be written as

$$-\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\partial}{\partial\beta_k}[\lambda_j^*(\alpha_j^*)^2]\frac{\psi_k(t)}{\gamma_k} = \int_0^T\int_0^T\left[\frac{\partial\bar{m}^*(x)}{\partial S(x)}\cdot \underset{\sim}{Q}^*(x,y) - \right.$$

$$\left. - [\bar{r}(x)-\bar{m}^*(x)]\cdot\frac{\partial \underset{\sim}{Q}^*(x,y)}{\partial S(x)}\right][\bar{r}(y)-\bar{m}^*(y)]R_s(t,x)\,dx\,dy. \tag{67}$$

For this case, the optimum estimate of $S(t)$ is given by the expression

$$S^*(t) = \int_0^T R_s(t,x)\int_0^T\left[Tr\left[\underset{\sim}{R}^*(x,y)\frac{\partial \underset{\sim}{Q}^*(y,x)}{\partial S(x)}\right]+\left[\frac{\partial\bar{m}^*(x)}{\partial S(x)}\cdot \underset{\sim}{Q}^*(x,y) - \right.\right.$$

$$\left.\left. - [\bar{r}(x)-\bar{m}^*(x)]\cdot\frac{\partial \underset{\sim}{Q}^*(x,y)}{\partial S(x)}\right][\bar{r}(y)-\bar{m}^*(y)]\right]dy\,dx. \tag{68}$$

In addition, if the covariance function matrix $\underset{\sim}{R}(x,y)$ is independent of the signal transmitted, the estimate $S^*(t)$ is given as

$$S^*(t) = \int_0^T R_s(t,x)\int_0^T\frac{\partial\bar{m}^*(x)}{\partial S(x)}\cdot \underset{\sim}{Q}(x,y)\;[\bar{r}(y)-\bar{m}^*(y)]\,dy\,dx, \tag{69}$$

while if the vector mean $\bar{m}(x)$ is identically equal to zero the estimate $S^*(t)$ can be written

$$S^*(t) = \int_0^T R_s(t,x)\int_0^T\left[Tr\left[\underset{\sim}{R}^*(x,y)\frac{\partial \underset{\sim}{Q}^*(y,x)}{\partial S(x)}\right] - \bar{r}(x)\cdot\frac{\partial \underset{\sim}{Q}^*(x,y)}{\partial S(x)}\bar{r}(y)\right]dy\,dx. \tag{70}$$

If the inverse covariance function matrix or the vector mean have an implicit dependence on the intelligence-carrying signal, equations (66) through (70) do not apply but equations (63) through (65) are still valid. Expressions similar to equations (66) through (70) can still be obtained for particular problems but the form for these expressions would change from problem to problem.

As an example of an estimation problem, again consider the transmitted reference communications system shown in Figure 9. It is assumed that the intelligence-carrying signal $S(t)$ is a sample function of a Gaussian process with zero mean and covariance $R_s(t,x)$. The additive and multiplicative noises, m(t), $n_1(t)$ and $n_2(t)$, are again assumed to be sample functions of independent Gaussian processes with zero means and covariance functions as in equations (46) through (48).

The inverse covariance function matrix of the received waveforms assuming a particular realization of $S(t)$ is then, from equation (50), given as

$$\underset{\sim}{Q}(y,x) = \frac{2\,\delta(y-x)}{N_{01}N_{02} + M_0(N_{02}S(y)S(x) + N_{01})}\begin{bmatrix} M_0 + N_{02} & -M_0 S(x) \\ -M_0 S(y) & M_0 S(y) S(x) + N_{01} \end{bmatrix} . \qquad (71)$$

Note that this matrix only has an explicit dependence on the intelligence-carrying signal so that equations (66) through (70) can be used. The quantity $\frac{\partial \underset{\sim}{Q}^*(y,x)}{\partial S(x)}$ is then given as

$$\frac{\partial \underset{\sim}{Q}^*(y,x)}{\partial S(x)} = \frac{2\,\delta(y-x)\,M_0}{[N_{01}N_{02} + M_0(N_{02}S(y)S(x) + N_{01})]^2}\begin{bmatrix} -N_{02}S(y)(M_0 + N_{02}) & -N_{01}(M_0 + N_{02}) \\ M_0 N_{02} S^2(y) & M_0 N_{01} S(y) \end{bmatrix}, \qquad (72)$$

so that

$$\int_0^T Tr\left[\underset{\sim}{R}(x,y)\,\frac{\partial Q^*(y,x)}{\partial S(x)}\right] dy = \frac{-M_0 N_{02} S^*(x)}{N_{01}N_{02} + M_0\left[N_{02}[S^*(x)]^2 + N_{01}\right]} \qquad (73)$$

and

$$\int_0^T \overline{r}(x)\cdot\frac{\partial \underset{\sim}{Q}^*(x,y)}{\partial S(x)}\,\overline{r}(y)\,dy = \frac{-2M_0[N_{01}N_{02} + M_0[N_{01} - N_{02}[S^*(x)]^2]]\,r^{(1)}(x)\,r^{(2)}(x)}{[N_{01}N_{02} + M_0[N_{02}[S^*(x)]^2 + N_{01}]]^2} +$$

$$+\frac{2M_0^2 N_{01} S^*(x)\,[r^{(2)}(x)]^2 - 2M_0 N_{02} S^*(x)(M_0 + N_{02})\,[r^{(1)}(x)]^2}{[N_{01}N_{02} + M_0[N_{02}[S^*(x)]^2 + N_{01}]]^2} . \qquad (74)$$

Equations (73) and (74) are then substituted into equation (70) to obtain an expression for $S^*(t)$. It is to be remembered that all terms affixed with an asterisk depend upon the estimate $S^*(t)$.

SUMMARY

Two classical problems in statistical communication theory, the detection and estimation of intelligence-carrying signals, have been considered for the case of transmission through a multidimensional Gaussian random channel. This channel is characterized by the property that a deterministic signal results in a set of Gaussian (in general, correlated) received waveforms.

For the detection problem, a receiver was found which operates on the set of received waveforms and gives as outputs, voltages proportional to the likelihood functions of the possible transmitted signals. Two configurations for this receiver are mathematically described in equations (30), (31) and (32) and equations (36), (37) and (38). A modification to this receiver which results in outputs which are proportional to the a posteriori probabilities of the signals is described by equation (35). This theory was then used to find the optimum detector for FSK transmission through a particular type of fading channel. The optimum detector was also found for a transmitted reference communications system.

For the estimation problem, a receiver was found which has as its output a signal which is the maximum a posteriori estimate of the intelligence-carrying signal. Various forms of this receiver are described mathematically in equations (63) through (70). As an example, a mathematical expression was found for the optimum estimate of a Gaussian intelligence-carrying signal for the case of a transmitted reference communications system.

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APPENDIX

In this Appendix, equations (64) through (67) of the text will be derived. The notation of this Appendix, except for the omission of the superscripts*, will be identical to that of the text and use will be made of previously defined functions.

Referring to the definition of the trace of a matrix given in equation (4), and using the expansions for $\underset{\sim}{R}(x,y)$ and $\underset{\sim}{Q}(x,y)$ given in equations (12) and (14), we can write,

$$\int_0^T \int_0^T Tr\left[\underset{\sim}{R}(x,y)\frac{\partial \underset{\sim}{Q}(y,x)}{\partial \beta_k}\right] dy\, dx =$$

$$= \int_0^T \int_0^T \sum_{m=1}^{M} \sum_{n=1}^{M} \left[\sum_{i=1}^{\infty} \frac{\theta_{i,m}(x)\theta_{i,n}(y)}{\lambda_i}\right]\left[\sum_{j=1}^{\infty} \frac{\partial \lambda_j}{\partial \beta_k}\theta_{j,m}(x)\theta_{j,n}(y)\right] dx\, dy$$

$$+ \int_0^T \int_0^T \sum_{m=1}^{M} \sum_{n=1}^{M} \left[\sum_{i=1}^{\infty} \frac{\theta_{i,m}(x)\theta_{i,n}(y)}{\lambda_i}\right]\left[\sum_{j=1}^{\infty} \lambda_j\left(\frac{\partial \theta_{j,m}(x)}{\partial \beta_k}\theta_{j,n}(y) + \right.\right. \tag{75}$$

$$\left.\left. + \theta_{j,m}(x)\frac{\partial \theta_{j,n}(y)}{\partial \beta_k}\right)\right] dx\, dy.$$

Noting the orthonormality of the vectors $\bar{\theta}(x)$, this equation becomes

$$\int_0^T \int_0^T Tr\left[\underset{\sim}{R}(x,y)\frac{\partial \underset{\sim}{Q}(y,x)}{\partial \beta_k}\right] dx\, dy = \sum_{j=1}^{\infty} \frac{1}{\lambda_j}\frac{\partial \lambda_j}{\partial \beta_k} +$$

$$+ \int_0^T \sum_{m=1}^{M} \sum_{j=1}^{\infty} \theta_{j,m}(x)\frac{\partial \theta_{j,m}(x)}{\partial \beta_k} dx + \int_0^T \sum_{n=1}^{M} \sum_{j=1}^{\infty} \theta_{j,n}(y)\frac{\partial \theta_{j,n}(y)}{\partial \beta_k} dy. \tag{76}$$

The last two expressions on the right hand side of equation (76) are zero as can be seen by writing

$$\frac{\partial}{\partial \beta_k}\left[\int_0^T \sum_{n=1}^{M} \theta_{j,n}(x)\,\theta_{j,n}(x)\, dx = 1\right], \tag{77}$$

or

$$\int_0^T \sum_{n=1}^{M} \theta_{j,n}(x)\frac{\partial \theta_{j,n}(x)}{\partial \beta_k} dx = 0. \tag{78}$$

Multiplying the remaining terms in equation (76) by $\frac{1}{2}\frac{\psi_k(t)}{\gamma_k}$ and summing over the index k results in the expression

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k} = \sum_{k=1}^{\infty}\frac{1}{2}\int_0^T\int_0^T Tr\left[\underset{\sim}{R}(x,y)\frac{\partial\underset{\sim}{Q}(y,x)}{\partial\beta_k}\right]\frac{\psi_k(t)}{\gamma_k}\,dy\,dx \tag{79}$$

which is equation (64) of the text.

If $Q_{mn}(x,y)$ depends on the intelligence-carrying signal (and thus on β_k) only through the appearance of terms which are implicit functions of $S(x)$ and $S(y)$, then

$$\frac{\partial Q_{mn}(y,x)}{\partial\beta_k} = \frac{\partial Q_{mn}(y,x)}{\partial S(y)}\psi_k(y) + \frac{\partial Q_{mn}(y,x)}{\partial S(x)}\psi_k(x), \tag{80}$$

so that

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k} = \frac{1}{2}\int_0^T\int_0^T\sum_{m=1}^{M}\sum_{n=1}^{M}R_{mn}(x,y)\frac{\partial Q_{nm}(y,x)}{\partial S(y)}$$

$$\left[\sum_{k=1}^{\infty}\frac{\psi_k(y)\psi_k(t)}{\gamma_k}\right]dy\,dx + \frac{1}{2}\int_0^T\int_0^T\sum_{m=1}^{M}\sum_{n=1}^{M}R_{mn}(x,y)\frac{\partial Q_{nm}(y,x)}{\partial S(x)}\left[\sum_{k=1}^{\infty}\frac{\psi_k(x)\psi_k(t)}{\gamma_k}\right]dy\,dx. \tag{81}$$

Since $Q_{mn}(x,y) = Q_{nm}(y,x)$ and

$$R_s(t,x) = \sum_{k=1}^{\infty}\frac{\psi_k(t)\psi_k(x)}{\gamma_k} \tag{82}$$

we can finally write

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\lambda_j}\frac{\partial\lambda_j}{\partial\beta_k}\frac{\psi_k(t)}{\gamma_k} = \int_0^T\int_0^T R_s(t,x)\; Tr\left[\underset{\sim}{R}(x,y)\frac{\partial\underset{\sim}{Q}(y,x)}{\partial S(x)}\right]dx\,dy \tag{83}$$

which is equation (66) of the text.

From equations (53) and (55), we can write

$$\sum_{j=1}^{\infty}\lambda_j\alpha_j^2 = \int_0^T\int_0^T[\bar{r}(x) - \bar{m}(x)]\cdot\underset{\sim}{Q}(x,y)[\bar{r}(y) - \bar{m}(y)]\,dy\,dx. \tag{84}$$

Taking the partial derivative of both sides of equation (84) with respect to β_k multiplying by $\frac{\psi_k(t)}{\gamma_k}$ and summing over the index k results in the equation

$$-\frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{\partial}{\partial\beta_k}\lambda_j(a_j)^2\right]\frac{\psi_k(t)}{\gamma_k} =$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}\int_0^T\int_0^T\frac{\partial}{\partial\beta_k}\left[\bar{r}(x)-\bar{m}(x)\right]\cdot\underset{\sim}{Q}(x,y)\left[\bar{r}(y)-\bar{m}(y)\right]\frac{\psi_k(t)}{\gamma_k}\,dx\,dy \tag{85}$$

which is equation (65) of the text.

Let us restrict ourselves to the case where $\underset{\sim}{Q}(y,x)$ and $\bar{m}(x)$ depend on the intelligence-carrying signal only through implicit functions of $S(x)$ and $S(y)$.

Then equation (85) can be rewritten as

$$-\frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial}{\partial\beta_k}(\lambda_j a_j^2)\frac{\psi_k(t)}{\gamma_k} = \int_0^T\int_0^T\frac{\partial\bar{m}(x)}{\partial S(x)}\cdot\underset{\sim}{Q}(x,y)\left[\bar{r}(y)-\bar{m}(y)\right]\sum_{k=1}^{\infty}\frac{\psi_k(t)\psi_k(x)}{\gamma_k}\,dx\,dy \tag{86}$$

$$-\int_0^T\int_0^T\left[\bar{r}(x)-\bar{m}(x)\right]\cdot\frac{\partial\underset{\sim}{Q}(x,y)}{\partial S(x)}\left[\bar{r}(y)-\bar{m}(y)\right]\sum_{k=1}^{\infty}\frac{\psi_k(t)\psi_k(x)}{\gamma_k}\,dx\,dy.$$

Referring to equation (82), equation (86) becomes

$$-\frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial}{\partial\beta_k}(\lambda_j a_j^2)\frac{\psi_k(t)}{\gamma_k} =$$

$$=\int_0^T\int_0^T\left[\frac{\partial\bar{m}(x)}{\partial S(x)}\cdot\underset{\sim}{Q}(x,y)-\left[\bar{r}(x)-\bar{m}(x)\right]\cdot\frac{\partial\underset{\sim}{Q}(x,y)}{\partial S(x)}\left[\bar{r}(y)-\bar{m}(y)\right]\right]R_s(t,x)\,dx\,dy$$

which is just equation (67) of the text.

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Rome Air Development Center.
Griffiss Air Force Base, New York.
ON THE DETECTION AND ESTIMATION PROBLEM FOR MULTIDIMENSIONAL GAUSSIAN RANDOM CHANNELS by J.K. Wolf. Nov. 1961. 26 pp. incl. illus. (Project No.: 4519; Task No.: 451903)(RADC-TR-61-214).

(Unclassified Report)

Two problems in statistical communication theory, the detection and estimation of intelligence-carrying signals are considered for the case of transmission through a multidimensional Gaussian random channel. Such a channel is characterized by the property that a deterministic input results in a set of received waveforms which are sample functions of Gaussian processes.

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1. Statistical Analysis.
2. Statistical Decision Theory.
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